WAAC2024

LZ78 Substring Compression in CDAWG-compressed space

Hiroki Shibata[1], Dominik Köppl[2]

1. Kyushu University

2. University of Yamanashi

LZ78 Factorization [Ziv & Lempel, '78]

The LZ78 factorization of a string T is a factorization

$$T = F_0 F_1 \dots F_f$$

T' is the unfactorized part of T.

where F_i $(i \geq 1)$ is the longest prefix of $T' = F_i \dots F_f$ that can be represented by $F_i = F_j c$ $(j < i, c \in \Sigma)$.

$$T = \text{abaabaac} = F_0 F_1 \dots F_f$$

$$F = (\varepsilon, \text{a}, \text{b}, \text{aa}, \text{ba}, \text{ac})$$

F' is a compressed representation of T.

$$F' = ((0, \text{a}), (0, \text{b}), (1, \text{a}), (2, \text{a}), (1, \text{c}))$$

($F_0 = \varepsilon$ is the empty string)

Substring Compression Problem

- **The text T of length n is given in advance.**
  - We can construct an index of T before the queries.
- **Queries:**
  - Input: two integers l, r $(1 \leq l \leq r \leq n)$
  - Output: The compressed representation of $T[l..r]$.

We only consider the case $(l, r) = (1, n)$ in this presentation, but the algorithms can be extended for arbitrary (l, r).

T

→ The LZ78 factorization of $T[l..r]$ is $(\mathrm{b}, \mathrm{a}, \mathrm{ab}, \mathrm{aa})$.

Previous Work

We focus on LZ78 substring compression problem.

Recently, [Köppl, '21] proposed a method for LZ78 substring compression.

- **This method is time optimal but consumes $O(n)$ words of space.**

Time/Space Complexity

	Space for Index	Working Space	Preprocessing Time	Query Time
[Köppl, '21]	$O(n)$	$O(z_{l,r})$	$O(n)$	$O(z_{l,r})$

($z_{l,r}$ is the number of LZ78 factors of $T[l..r]$.)

Our Work

We propose a method for LZ78 substring compression in compressed space.

Time/Space Complexity

	Space for Index	Working Space	Preprocessing Time	Query Time
[Köppl, '21]	$O(n)$	$O(z_{l,r})$	$O(n)$	$O(z_{l,r})$
Ours	$O(e)$	$O(z_{l,r})$	$O(n)$	$O(z_{l,r} \log n)$

Our method is the space-efficient variant of [Köppl, '21], which will be recalled in the next slides.

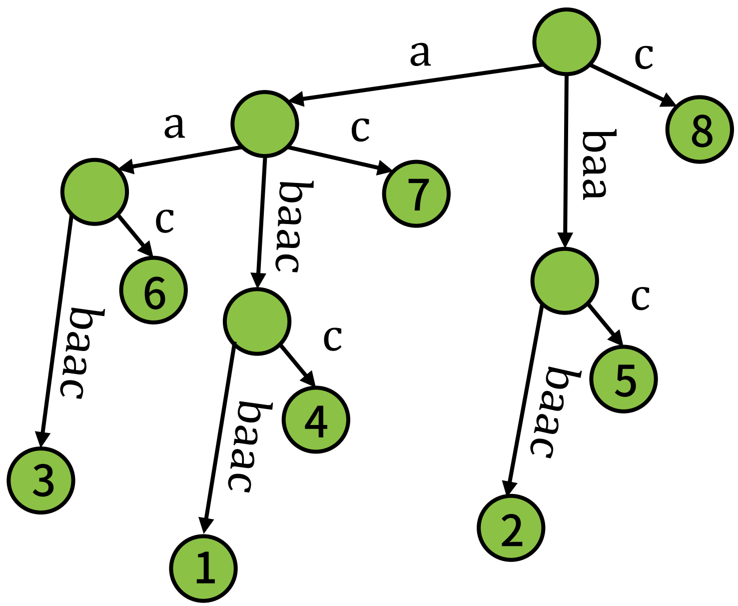
($z_{l,r}$ is the number of LZ78 factors of $T[l..r]$.)

(e is the number of the edges of the CDAWG of T.)

Computing LZ78 by suffix trees (previous method)

Suffix Trees (ST)

Suffix Tree: The compact trie representing all suffixes of T.



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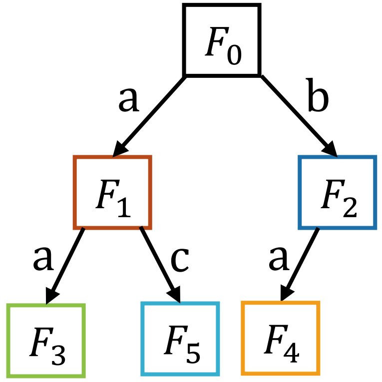
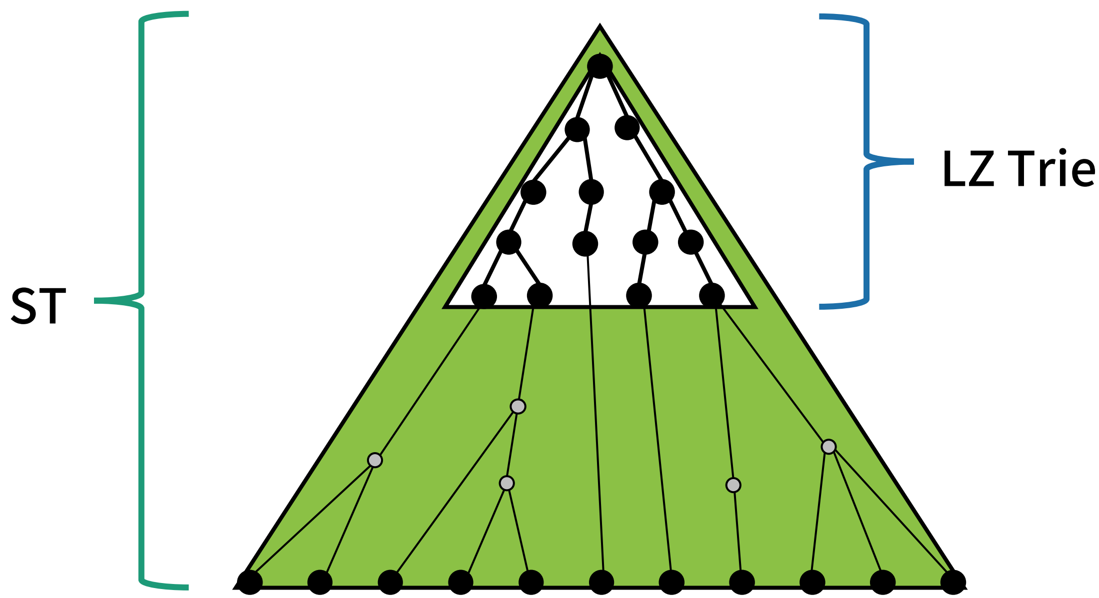


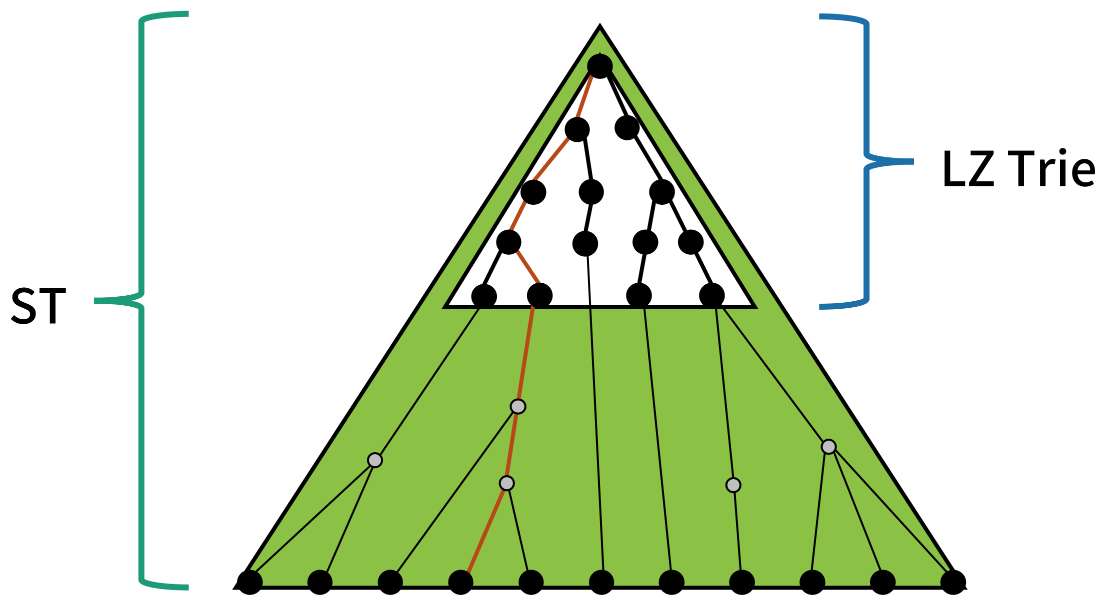
Superimposing LZ78 trie onto ST

- **We can superimpose the LZ78 trie onto the ST.**
  - The LZ trie is an induced subgraph over the ST consisting only of the LZ78 factors.



Computing LZ78 by ST

- The new factor can be computed as follows:
  1. Find the path representing the unfactorized part of T.
  2. Find the lowest LZ78 node on the path.
  3. Add an LZ78 node immediately below the lowest LZ78 node.



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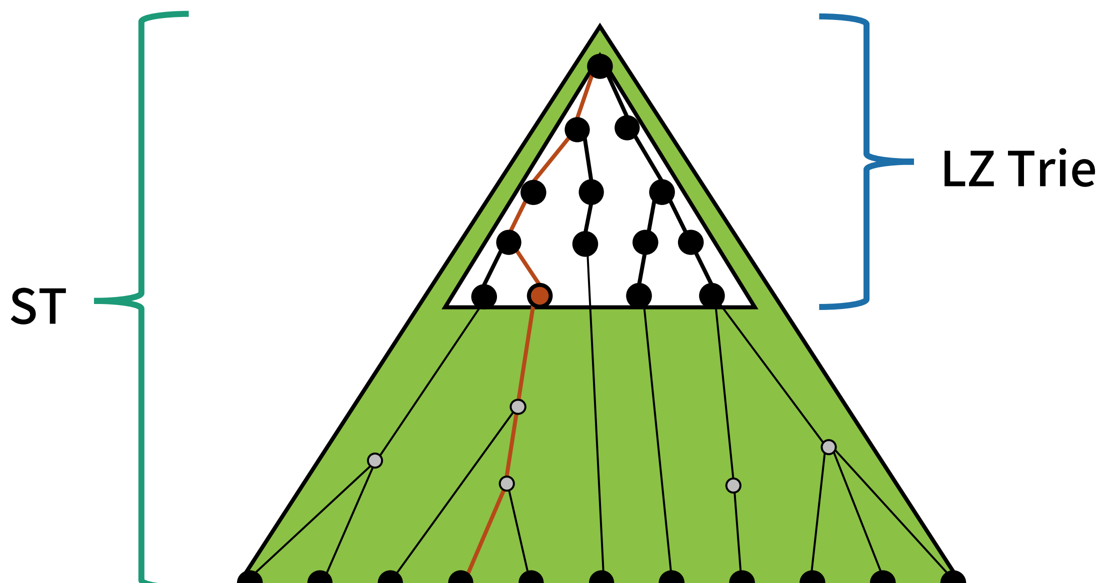

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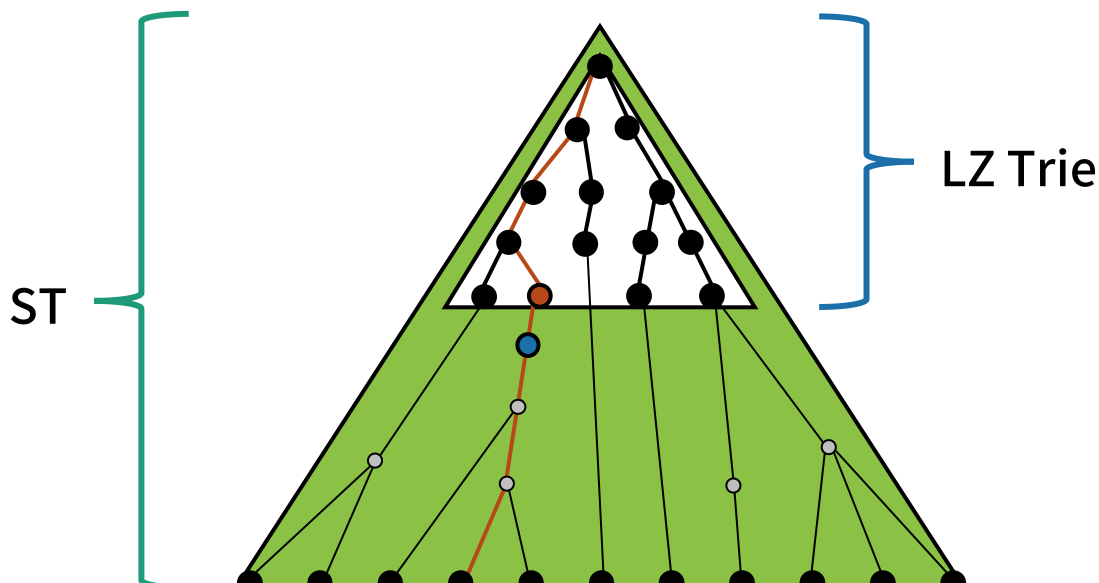


Implementation and Complexity

Required data structures:

- Lowest Marked Ancestor (LMA) [Westbrook, 1992]
  - amortized O(1) time / query
- Weighted Level Ancestor [Gawrychowski et al., 2014]
  - O(1) time / query

Time / Space complexity

- Each data structure can be stored in $O(n)$ words.
- We can find the lowest factor and add a factor in constant time per factor.

■ **Overall Complexity ⇒ $O(n)$ words • $O(z_{l,r})$ time / query**

($z_{l,r}$ is the number of LZ78 factors of $T[l..r]$)

Computing LZ78 in compressed space (proposed method)

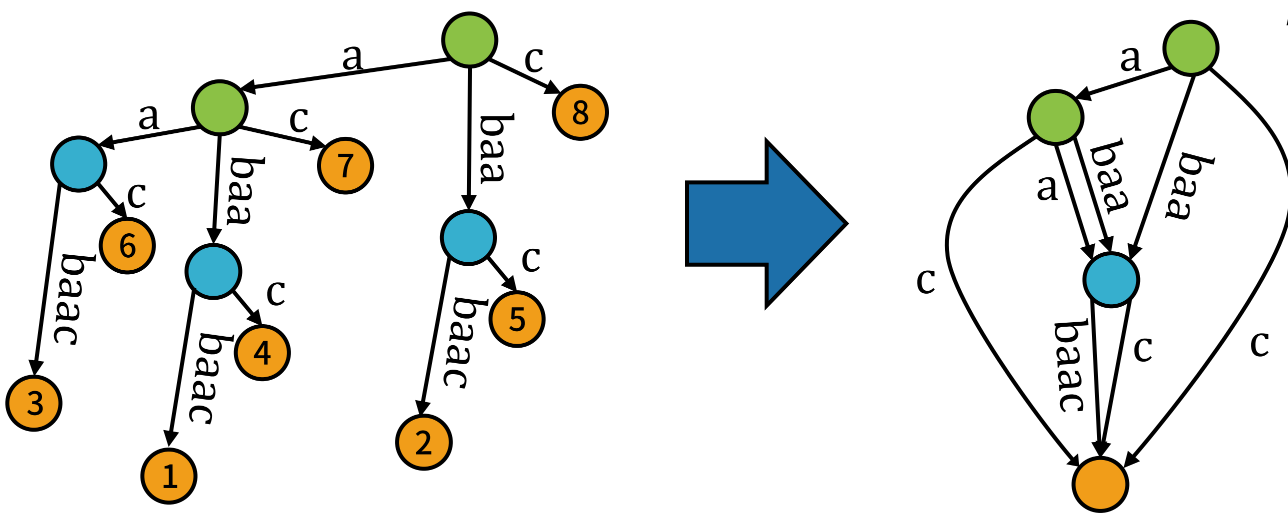
CDAWG (Compact Directed Acyclic Word Graph)

CDAWG: The edge-labelled DAG which obtained by merging isomorphic subtrees of the corresponding ST.

- **Property: The number e of edges of the CDAWG is small for some highly repetitive strings.**

  → CDAWG can be regarded as a compressed representation of STs.

$T = \text{abaabaac}$



Why CDAWG cannot simply replace ST?

Replacing ST with CDAWG is difficult because

1. several edges can end at the same node, and
2. a node of CDAWG can represent several strings.

So we use an additional data structure.

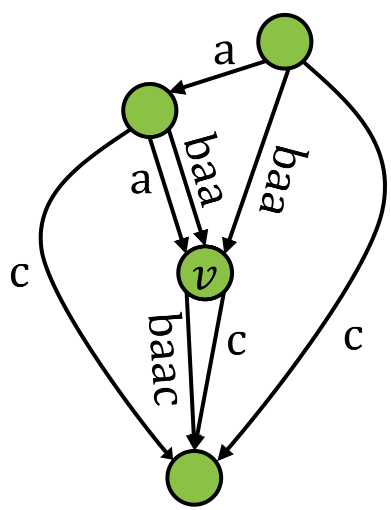


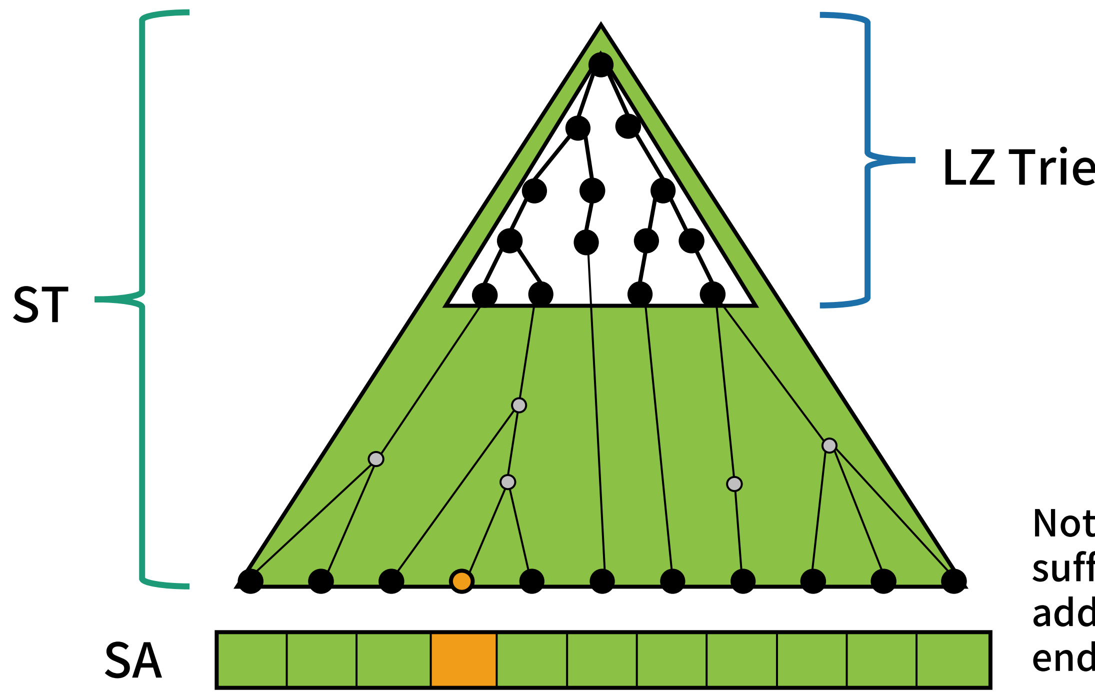
Fig: The CDAWG of $T = \mathrm{abaabaac}$.

- v represents abaa, baa, and ba.
- The indegree of v is 3.

Suffix Arrays (SA)

The suffix array stores the lexicographic order of all suffixes.

- Since the leaves have a one-to-one correspondence to all suffixes, the SA represents the order of the leaves.
- One leaf of the ST corresponds to one position on the SA.

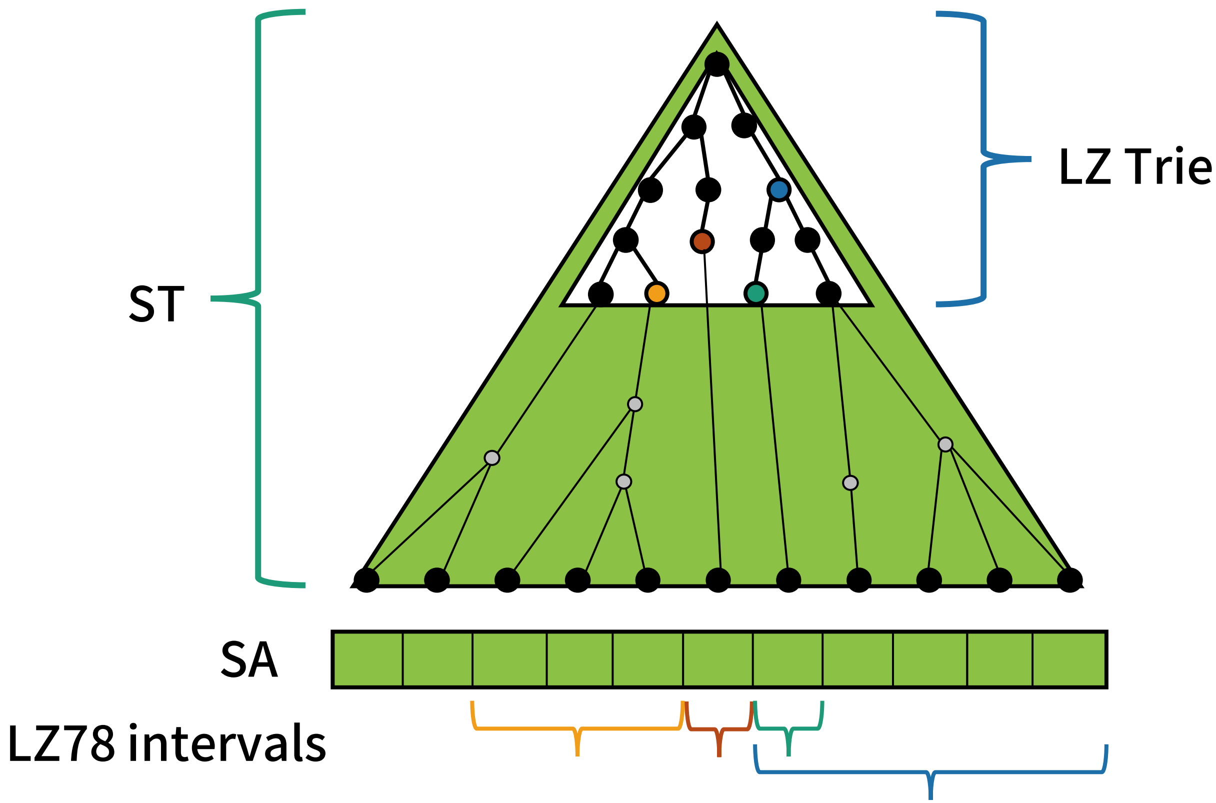


Note: We can guarantee that each suffix corresponds to a leaf node by adding a unique character at the end of T.

Converting LZ78 Nodes to Intervals

Each LZ78 node corresponds to an interval on SA.

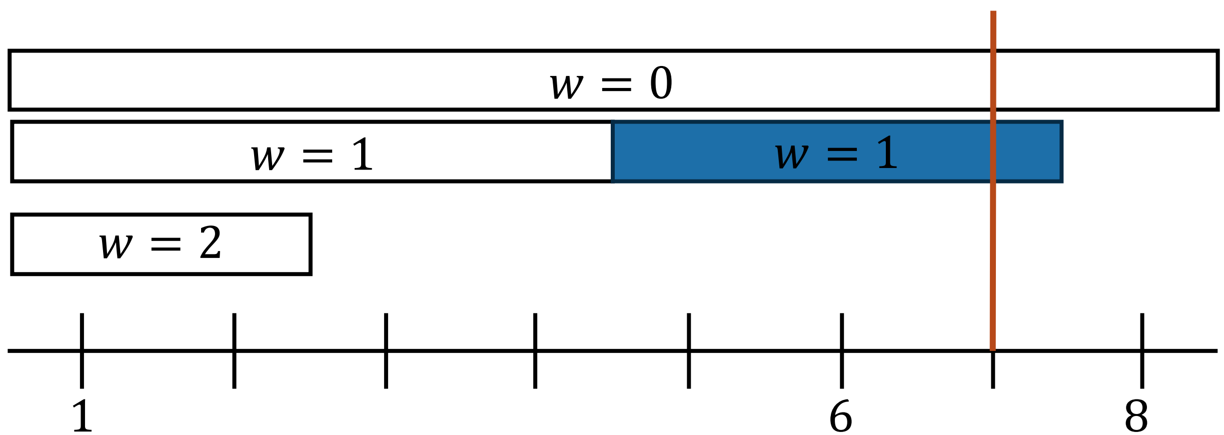
→ The set of all LZ78 nodes can be represented by the set of intervals.



Stabbing-Max Problem

Finding lowest LZ78 node can be reduced to the following queries and operations:

1. **Find the interval $[a, b] \in S$ containing k whose weight is maximum.**
   - Corresponding to find the lowest LZ node.
2. **Add an interval $[a, b]$ with weight w to a set S.**
   - Corresponding to add an LZ node.



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Finding lowest LZ78 node can be reduced to the following queries and operations:

1. Find the interval $[a, b] \in S$ containing k whose weight is maximum.
2. Add an interval $[a, b]$ with weight w to a set S.

This problem is known as the stabbing-max problem, and there is a data structure that performs any query with the following complexity [Tarjan, 1979]:

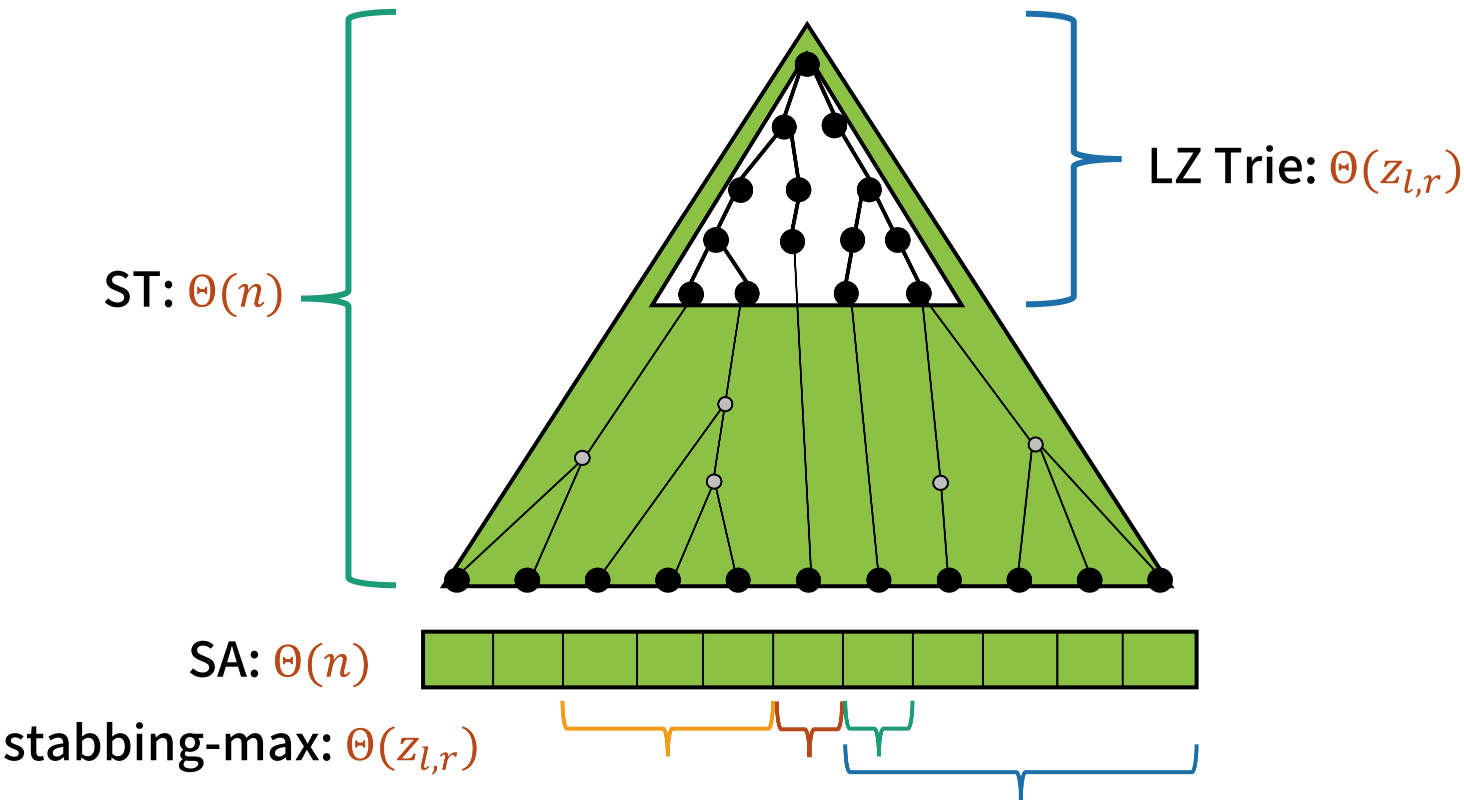
- Add/Find an interval: $O(\log m)$ time / query
- Space complexity: $O(m)$ words

(m is the number of intervals)

Bottoleneck of Space Complexity

Now, our data structure uses $\Theta(n)$ space because of the ST and the SA.

To reduce the space, we need to compute LZ78 factors without them.

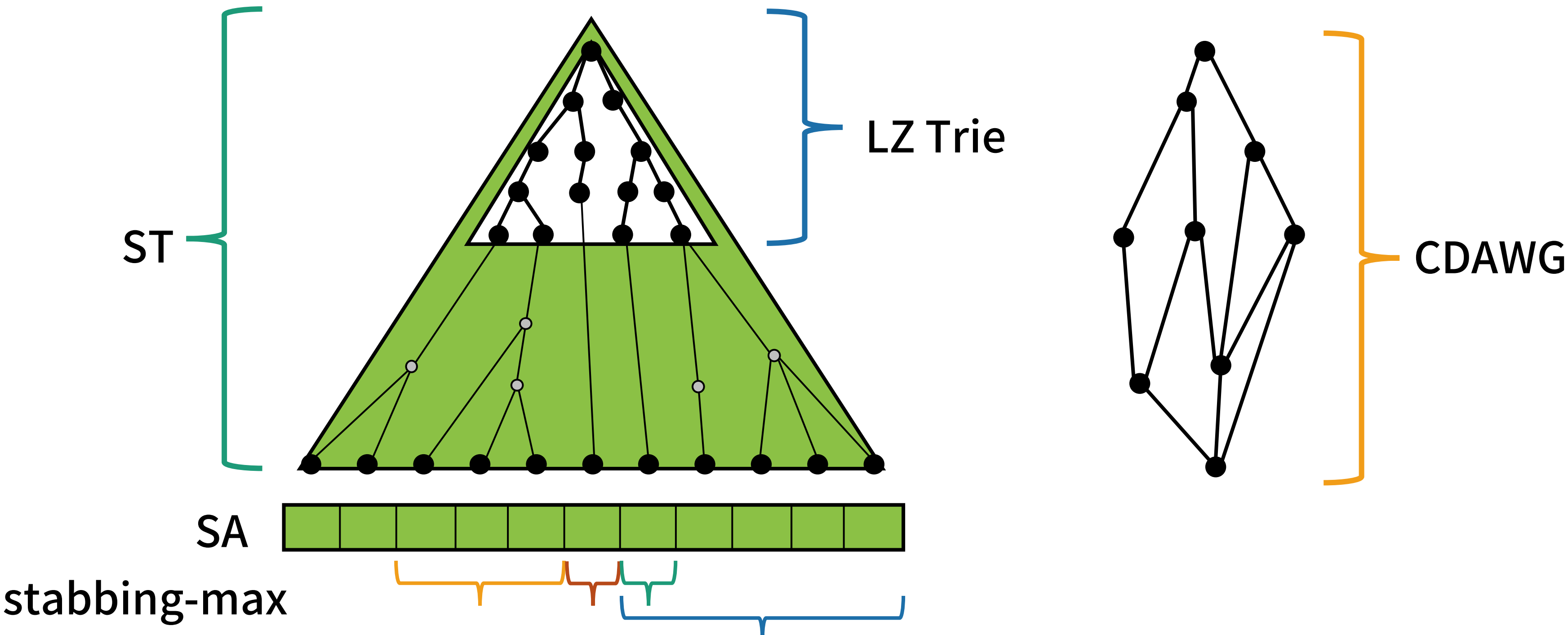


Note: $z_{l,r} \in O(n)$

Replacing ST/SA with CDAWG

Since CDAWG is a compacted variant of ST, some CDAWG-based index efficiently performs ST/SA operation.

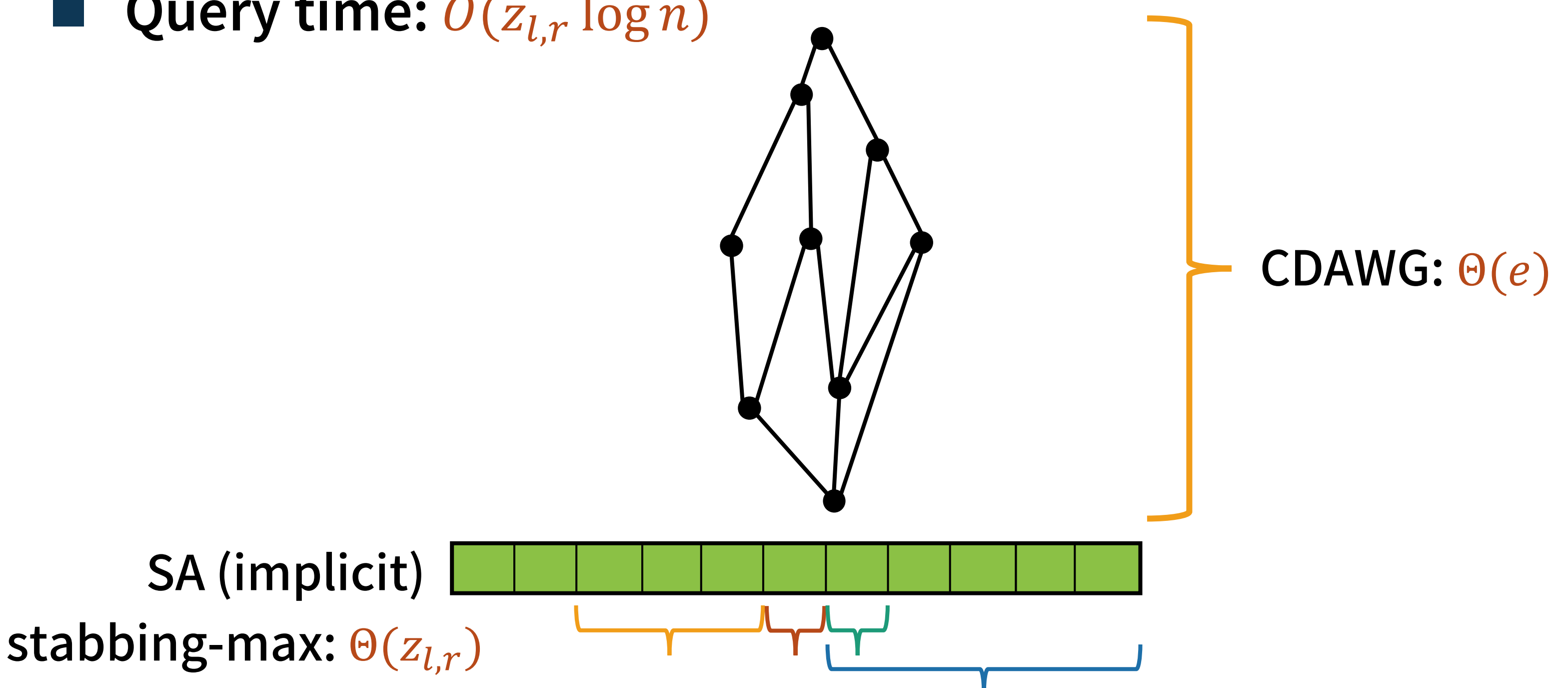
There is a CDAWG-based index that performs ST/SA operation in $O(\log n)$ **time** and is stored in $O(e)$ **space**. [Bealazzougui and Cunial, CPM 2017]



Overall Structure

Our method only uses the CDAWG and the stabbing-max structure.

- Space for index: $\Theta(e)$ words
- Working space for queries: $\Theta(z_{l,r})$ words
- Query time: $O(z_{l,r} \log n)$



Conclusion

- We proposed a method for LZ78 substring compression in compressed space.
- Some applications:
  - We can replace the CDAWG by some other compressed index.
  - Applying this method for other LZ78-like compressions. (LZD, LZMW)

Time/Space Complexity

	Space for Index	Working Space	Preprocessing Time	Query Time
[Köppl, '21]	$O(n)$	$O(z_{l,r})$	$O(n)$	$O(z_{l,r})$
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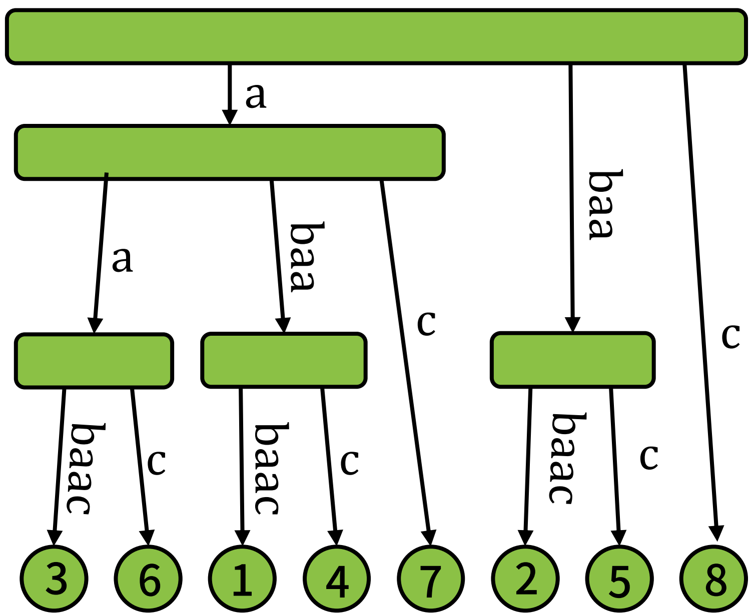
Suffix Tree: The compact trie representing all suffixes of T.



$T = \text{abaabaac}$

LZ78 Factorization [Ziv & Lempel, '78]

- One of a text compression method.
- Factorize a string T to $T = F_0F_1 \dots F_f$ by a specific algorithm.

$$T = \text{abbabaaab} = F_0F_1 \dots F_f$$

$$F = (\varepsilon, \text{a}, \text{b}, \text{ba}, \text{baa}, \text{ab})$$

($F_0 = \varepsilon$ is the empty string)

LZ78 Factorization [Ziv & Lempel, '78]

- F_i $(i \geq 1)$ **is computed as follows:**
  1. Compute the longest prefix of $T' = T[1 + |F_0| + |F_1| + \cdots + |F_{i-1}|..n]$ that appears in F.
  2. Set $F_i = F_j T'[\ell + 1]$, where F_j is the longest factor that precedes T'.

$$T = \mathrm{abbabaaab} = F_0 F_1 \ldots F_f \; (F_0 = \varepsilon)$$

$$T' = \mathrm{abbabaaab}$$

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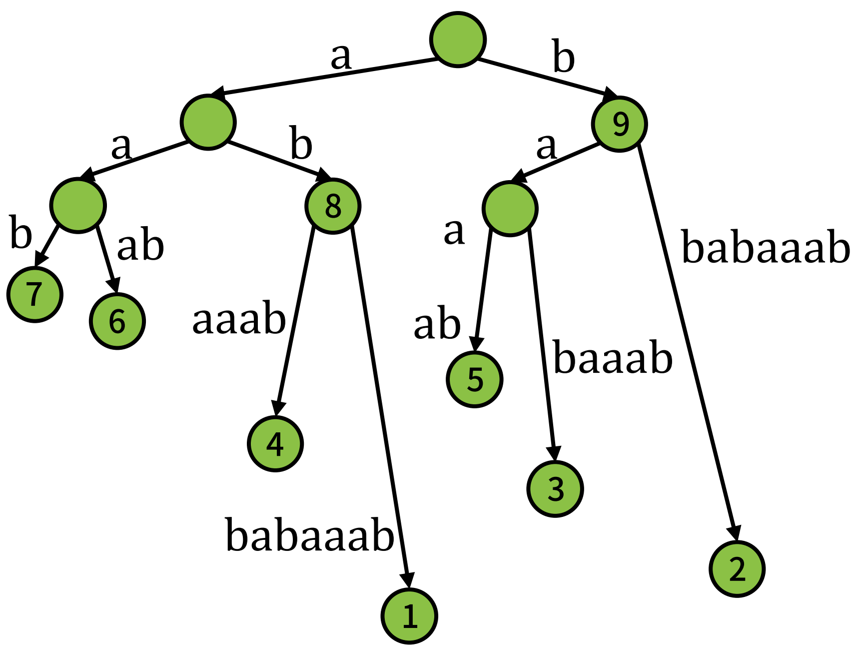

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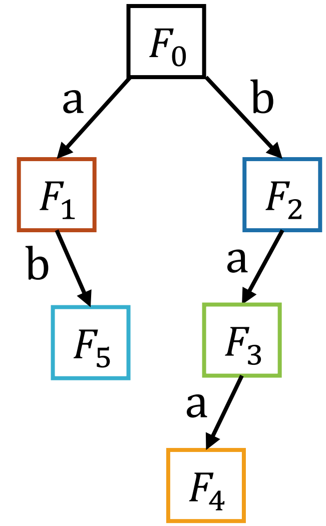

Computing LZ78 by ST

- F_i $(i \geq 1)$ **can be computed as follows:**
  1. Find the path on the ST representing $T' = T[1 + |F_0| + |F_1| + \cdots, |F_{i-1}|..n]$.
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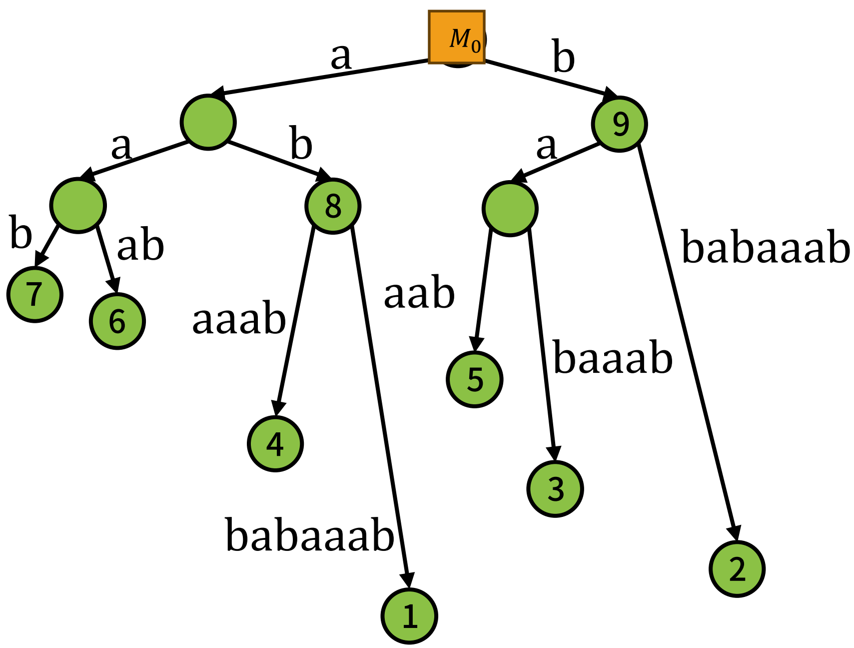

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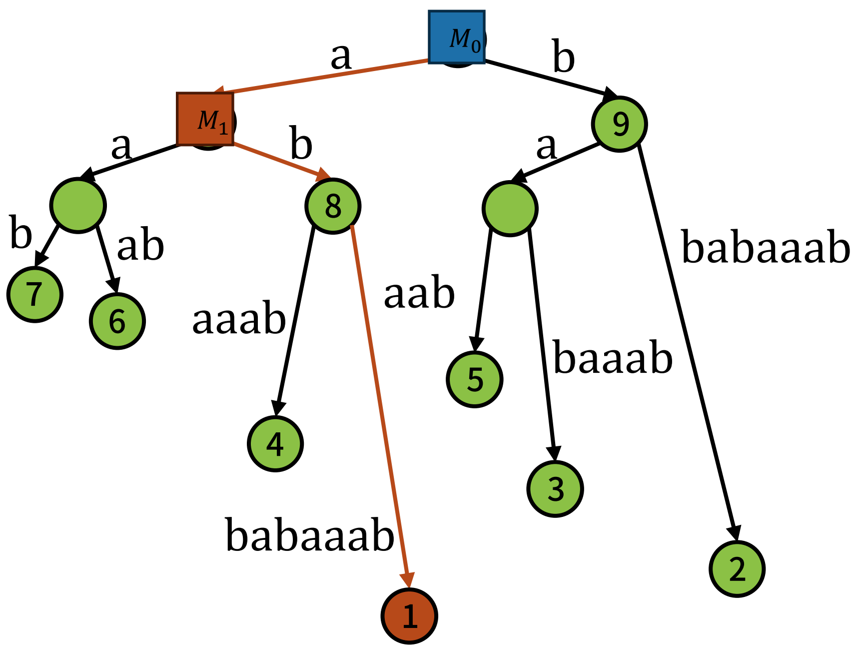

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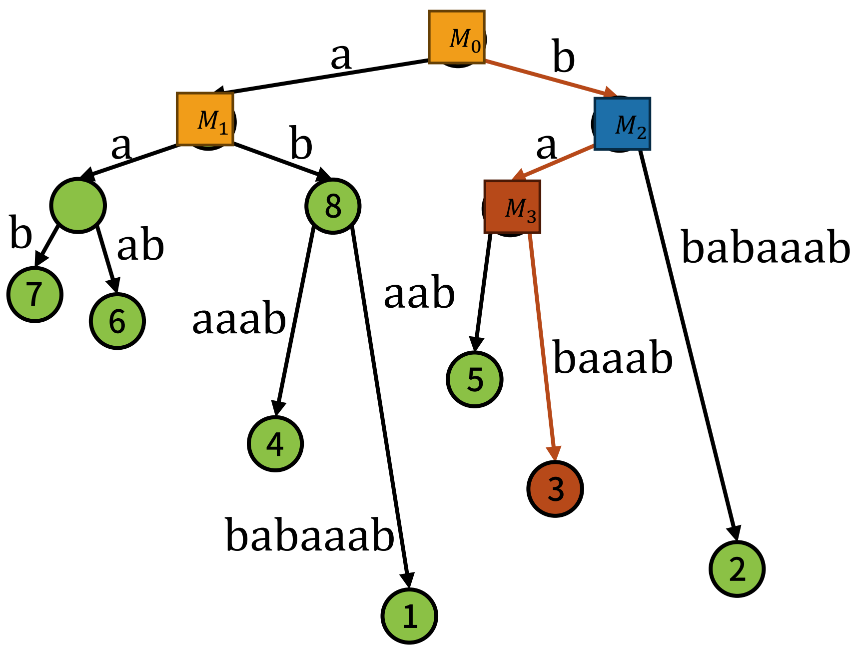

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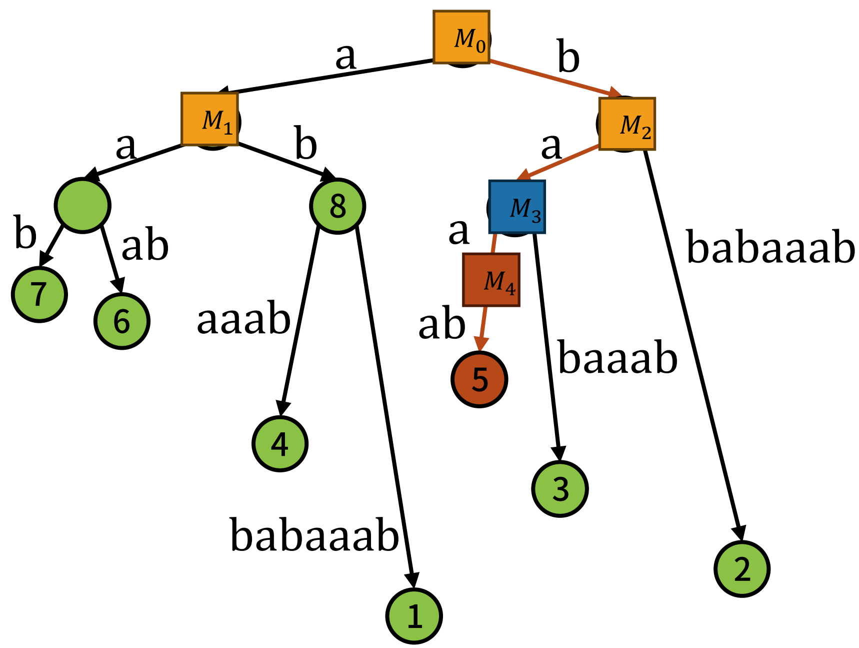

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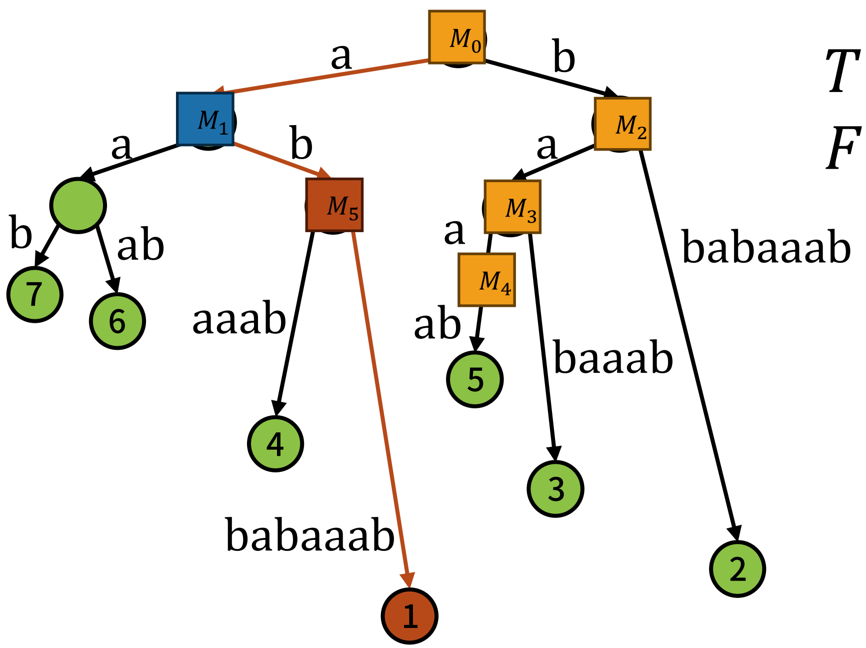

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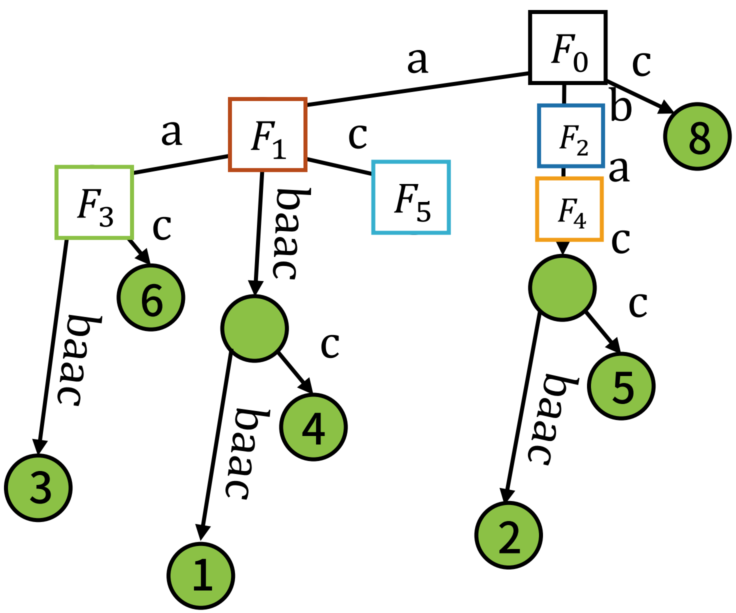

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