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Relaxation of Square-Freeness

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slides

Outline

1. **Background and Our Results**
2. Constructing Ternary 3^+-p-Square-Free Words
3. Constructing Binary 3^+-op-Square-Free Words
4. Conclusion

Squares in Words

A **square** is a non-empty word repeated twice in a row.

$$uu = \underbrace{\mathrm{abc}}_{u}\underbrace{\mathrm{abc}}_{u}$$

- A (possibly infinite) word is **square-free** if none of its subwords is a square.
- Thue showed that an infinite square-free word exists over a three-character alphabet. [Thue, 1906]
- No infinite binary word is square-free.
  - Every sufficiently long binary word contains one of aa, bb, abab, or baba.

Squares Under Equivalence Relations

We consider notions of equality based on **equivalence relations**.

- **Parameterized equivalence**: u and v are equivalent if there exists a character-wise bijection f such that $f(u) = v$.
  - **Example:** $u = \mathrm{abca}$ and $v = \mathrm{xyzx}$ are parameterized equivalent via $f : (\mathrm{a}, \mathrm{b}, \mathrm{c}) \mapsto (\mathrm{x}, \mathrm{y}, \mathrm{z})$.
- **Order-preserving equivalence**: u and v are equivalent if $u[i] \preceq u[j] \Leftrightarrow v[i] \preceq v[j]$ for all i, j.
  - **Example:** 1243 and 3586 are order-preserving equivalent.
- Since all one-letter words are equivalent under both relations, every two-letter word is a square.

Ignoring Short Squares

We consider avoiding **long squares**.

- **ℓ^+-square**: A square of length at least 2ℓ.
- We analyze the existence of infinite square-free words under a specific equivalence relation in terms of two parameters: σ (alphabet size) and ℓ (square length).
- In fact, these cases have been fully characterized [Ng et al., 2019], but the characterization relies on `Walnut` or a large morphism.

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Our Contribution

We give combinatorial constructions of infinite ℓ^+-square-free words under parameterized and order-preserving equivalence for specific pairs (σ, ℓ).

Our Contributions

Main results

1. We construct an infinite 3^+-parameterized-square-free ternary word.
2. We construct an infinite 3^+-order-preserving-square-free binary word.

- The main arguments are combinatorial; only a bounded set of short subwords is checked directly.

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A Square-Free Morphism

A **morphism** is a map that replaces each character with a fixed word.

The Morphism φ

$$\varphi(\mathrm{a}) = \mathrm{abc}, \qquad \varphi(\mathrm{b}) = \mathrm{ac}, \qquad \varphi(\mathrm{c}) = \mathrm{b}.$$

$$\begin{aligned} \varphi^0(\mathrm{a}) &= \mathrm{a}, \\ \varphi^1(\mathrm{a}) &= \mathrm{abc}, \\ \varphi^2(\mathrm{a}) &= \mathrm{abcacb}, \\ \varphi^3(\mathrm{a}) &= \mathrm{abcacbabcbac}. \end{aligned}$$

Every $\varphi^k(\mathrm{a})$ is square-free [Cori and Formisano, 1990].

Word Construction

Let $C = \mathrm{cbbbc}$ and define

$$\mu(\mathrm{a}) = \mathrm{a}, \qquad \mu(\mathrm{b}) = \mathrm{b}, \qquad \mu(\mathrm{c}) = C.$$

For every $k \geq 0$, let

$$x_k = \varphi^k(\mathrm{a}) \qquad \text{and} \qquad y_k = \mu(x_k).$$

For example,

$$x_3 = \mathrm{ab\,c\,a\,c\,bab\,c\,ba\,c}$$

$$y_3 = \mathrm{ab\,cbbbc\,a\,cbbbc\,bab\,cbbbc\,ba\,cbbbc}.$$

Goal

Show that every y_k is 3^+-parameterized-square-free.

Proof Outline

Goal

Show that every y_k is 3^+-parameterized-square-free.

We prove two statements about y_k.

Proposition 1: Parameterized squares become ordinary squares

If a subword w of y_k is a 3^+-parameterized square, then w is an ordinary square.

Proposition 2: No long ordinary square

y_k is 3^+-square-free.

Combining these results shows that y_k is 3^+-parameterized-square-free.

c-Free Subwords of x_k

Consider a subword of x_k avoiding c.

- Since x_k is square-free, this subword is a square-free binary word.
- It contains none of aa, bb, abab, or baba and has length at most 3.
- aba does not appear in x_k, because no concatenation of abc, ac, and b produces it.

Word Construction

$$\varphi : (\mathrm{a}, \mathrm{b}, \mathrm{c}) \mapsto (\mathrm{abc}, \mathrm{ac}, \mathrm{b})$$

$$\mu : (\mathrm{a}, \mathrm{b}, \mathrm{c}) \mapsto (\mathrm{a}, \mathrm{b}, \mathrm{cbbbc})$$

$$x_k = \varphi^k(\mathrm{a})$$
$$y_k = \mu(x_k)$$

c-Free Subwords of x_k

$$S = \{\mathrm{a}, \mathrm{b}, \mathrm{ab}, \mathrm{ba}, \mathrm{bab}\}.$$

bb-Free Subwords of y_k

Lemma 1

If a subword w of y_k is bb-free and $|w| \geq 6$, then w is not a parameterized square.

- y_k can be factorized as $s_1 C s_2 C s_3 \cdots$, where $s_i \in S$.
- w does not contain any length-3 subwords of C.
  (because such subword contains bb)

$$x_3 = \mathrm{ab} \mid \mathrm{c} \mid \mathrm{a} \mid \mathrm{c} \mid \mathrm{bab} \mid \mathrm{c} \mid \mathrm{ba} \mid \mathrm{c}$$

$$y_3 = \mathrm{ab} \mid \mathrm{cb}\,\underbrace{\mathrm{bbc} \mid \mathrm{a} \mid \mathrm{cb}}_{\text{contains bb}}\,\mathrm{bbc} \mid \mathrm{bab} \mid \mathrm{cbb}\,\underbrace{\mathrm{bc} \mid \mathrm{ba} \mid \mathrm{cb}}_{\text{bb-free}}\,\mathrm{bbc}$$

bb-Free Subwords of y_k

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- w does not contain any length-3 subwords of C.

Structure of a Subword Avoiding bb

The subword w can be written as

$$w = \alpha s \beta, \qquad s \in S,$$

where $s \in S$, α is a suffix of C with $|\alpha| \leq 2$, and β is a prefix of C with $|\beta| \leq 2$.

bb-Free Subwords of y_k

Structure of a Subword Avoiding bb

The subword w can be written as

$$w = \alpha s \beta, \qquad s \in S,$$

where $s \in S$, α is a suffix of C with $|\alpha| \leq 2$, and β is a prefix of C with $|\beta| \leq 2$.

- Since $|\alpha|, |\beta| \leq 2$ and $|w| \geq 6$, we have $|s| \geq 2$. Hence, $s \in \{\mathrm{ab}, \mathrm{ba}, \mathrm{bab}\}$.
- The even-length candidates are bcbabc, cbabcb, bcabcb, and bcbacb; none is a parameterized square.

Thus, if a subword w of y_k is bb-free and $|w| \geq 6$, then w is not a parameterized square.

Proposition 1: Reduction to Equality

Proposition 1: Parameterized squares become ordinary squares

If a subword w of y_k is a 3^+-parameterized square, then w is an ordinary square.

Proof: Write $w = uv$ with $|u| = |v| \geq 3$ and $f(u) = v$ for a bijection f.

1. By Lemma 1, w contains bb.
2. Every occurrence of bb in y_k lies inside $C = \mathrm{cbbbc}$. Hence, one of u and v contains cbb or bbc.
3. The only subword of y_k parameterized equivalent to cbb (resp. bbc) is cbb (resp. bbc) itself. Thus, $f(\mathrm{b}) = \mathrm{b}$ and $f(\mathrm{c}) = \mathrm{c}$.
4. Since f is a bijection on $\{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$, it also fixes a. Thus, f is the identity and $u = v$.

Proposition 2: Offsets in C

Proposition 2: No long ordinary square

y_k is 3^+-square-free.

Recall that we can factorize y_k into

$$y_k = s_1 \, C \, s_2 \, C \, s_3 \, \cdots \, , \qquad s_i \in S, \qquad C = \mathrm{cbbbc}.$$

- A position in an occurrence of C has **offset** $r \in \{0, 1, 2, 3, 4\}$, measured from the beginning of C.
- Positions outside occurrences of C have no offset.

$$y_3 = \mathrm{ab} \mid \overset{0\,1\,2\,3\,4}{\mathrm{cbbbc}} \mid \mathrm{a} \mid \overset{0\,1\,2\,3\,4}{\mathrm{cbbbc}} \mid \mathrm{bab} \mid \overset{0\,1\,2\,3\,4}{\mathrm{cbbbc}} \mid \mathrm{ba} \mid \overset{0\,1\,2\,3\,4}{\mathrm{cbbbc}}.$$

Lemma 2 (1/3): Common Offsets

Lemma 2

For every subword u of y_k with $|u| \geq 3$, all occurrences of u start outside occurrences of C, or they all start at a common offset in occurrences of C.

We consider the five possible offsets $r = 0, 1, 2, 3, 4$ of an occurrence of u starting in an occurrence of C.

$$y_3 = \mathrm{ab} \mid \overset{0\,1\,2\,3\,4}{\mathrm{cbbbc}} \mid \mathrm{a} \mid \mathrm{cb}\underbrace{\overset{0\,1\,2\,3\,4}{\mathrm{bbc}} \mid \mathrm{ba}}_{u\ (r=2)}\mathrm{b} \mid \mathrm{cb}\underbrace{\overset{0\,1\,2\,3\,4}{\mathrm{bbc}} \mid \mathrm{ba}}_{u\ (r=2)} \mid \overset{0\,1\,2\,3\,4}{\mathrm{cbbbc}}.$$

Lemma 2 (2/3): Offsets 0, 1, and 2

Suppose that an occurrence of u has offset r.

- $r = 0$: u starts with cbb
- $r = 1$: u starts with bbb
- $r = 2$: u starts with bbc

Each prefix occurs only at its indicated offset in C. Thus, every occurrence of u starts at the same offset.

$$y_3 = \mathrm{ab} \mid \overset{0\,1\,2\,3\,4}{\underbrace{\mathrm{cbb}}_{r=0}\mathrm{bc}} \mid \mathrm{a} \mid \overset{0\,1\,2\,3\,4}{\mathrm{c}\underbrace{\mathrm{bbb}}_{r=1}\mathrm{c}} \mid \mathrm{bab} \mid \overset{0\,1\,2\,3\,4}{\mathrm{cb}\underbrace{\mathrm{bbc}}_{r=2}} \mid \mathrm{ba} \mid \overset{0\,1\,2\,3\,4}{\mathrm{cbbbc}}$$

Lemma 2 (3/3): Offsets 3 and 4

- $r = 3$: u starts with bc. The same prefix also occurs in $\mathrm{b}C$. This occurrence has no offset.
- $r = 4$: u starts with c. The same prefix also occurs at the start of C. This occurrence has offset 0.

The occurrences in $\mathrm{b}C$ and at offset 0 are followed by bb. Those at offsets 3 and 4 are not. Hence, u cannot occur at both positions[1].

$$y_3 = \mathrm{ab} \mid \overset{0\,1\,2\;\;3\,4}{\mathrm{cbb}\underbrace{\mathrm{bc}}_{\substack{r=3\\ \text{(not followed by bb)}}}} \mid \mathrm{a} \mid \overset{0\,1\,2\,3\,4}{\mathrm{cbbbc}} \mid \mathrm{ba}\underbrace{\mathrm{b} \mid}_{\substack{\text{no offset}\\ \text{(followed by bb)}}} \overset{0\,1\,2\,3\,4}{\mathrm{cbbbc}} \mid \mathrm{ba} \mid \overset{0\,1\,2\,3\,4}{\mathrm{cbbbc}}.$$

[1] The case $r = 3$ and $|u| = 3$ is exceptional. It can also be excluded.

Proposition 2: Proof Outline

Now we prove Proposition 2.

Proposition 2: No long ordinary square

y_k is 3^+-square-free.

Proof outline. Assume that a subword $w = uu$ $(|u| \geq 3)$ is a subword of y_k.

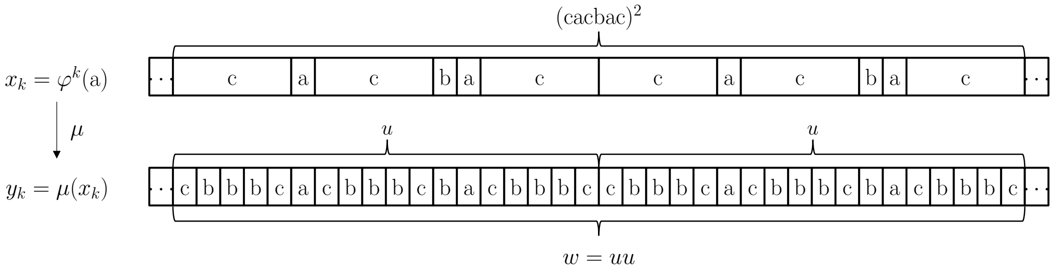
1. By Lemma 2, the two copies of u start outside occurrences of C, or at a common offset in C.
2. If they start outside C or at offset 0, their preimages under μ form a square in x_k.
3. At a positive offset, extending both copies to complete occurrences of C again gives a square in x_k.

Proposition 2: Outside C or at Offset 0

1. If both copies start outside C, or both have offset 0, each copy starts at the first character of $\mu(x)$ for some letter x of x_k.
2. We can apply μ^{-1} to u and obtain the preimage v:

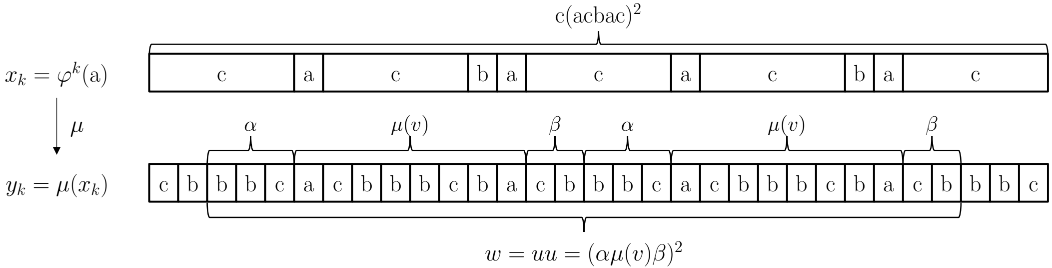
$$u = \mu(v), \qquad w = uu = \mu(v^2).$$

3. Thus, v^2 occurs in x_k, a contradiction to square-freeness of x_k.



Proposition 2: Positive Offsets

1. Since both copies start at the same offset, we can write $u = \alpha\mu(v)\beta$, where $\beta\alpha = C$.
2. Prepending β to w and appending α gives
$$\beta w \alpha = C\mu(v)C\mu(v)C = \mu(\mathrm{c}(v\mathrm{c})^2).$$
3. Thus, $(v\mathrm{c})^2$ occurs in x_k, a contradiction to square-freeness of x_k.



Proving the Theorem

Theorem 1: 3^+-Parameterized-Square-Free Ternary Words

For every $k \geq 0$, the word $y_k = \mu\left(\varphi^k(\mathrm{a})\right)$ is 3^+-parameterized-square-free.

1. By Proposition 1, every 3^+-parameterized square in y_k is an ordinary square uu with $|u| \geq 3$.
2. Proposition 2 excludes every such ordinary square, so y_k is 3^+-parameterized-square-free.

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Word Construction

Define a morphism $\tau : \Sigma_3^* \to \Sigma_2^*$ by

$$\tau(\mathrm{a}) = 00, \qquad \tau(\mathrm{b}) = 0101, \qquad \tau(\mathrm{c}) = 11.$$

For every $k \geq 0$, define

$$z_k = \tau(x_k) = \tau\big(\varphi^k(\mathrm{a})\big).$$

For example,

$$x_3 = \mathrm{a\,b\,c\,a\,c\,b\,a\,b\,c\,b\,a\,c}$$

$$z_3 = 00\,0101\,11\,00\,11\,0101\,00\,0101\,11\,0101\,00\,11.$$

Goal

Show that every z_k is 3^+-order-preserving-square-free.

Proposition 3: Square-Freeness

Proposition 3: Binary Images of Square-Free Words

If a ternary word x is square-free, then $\tau(x)$ is 3^+-square-free.

Proof outline (similar to the previous proof).
Assume that a subword u of $\tau(x)$ is a 3^+-square.

1. We can exclude short u by enumerating and checking all candidates.
2. For long u, the two copies must have the same offset value.
3. Applying the inverse morphism to a square subword gives a square subword of x, yielding a contradiction.

Theorem 2: Proof Outline

Theorem 2: 3^+-Order-Preserving-Square-Free Binary Words

For every $k \geq 0$, the word $z_k = \tau\big(\varphi^k(\mathrm{a})\big)$ is 3^+-order-preserving-square-free.

1. Suppose uv is an order-preserving square with $|u| = |v| \geq 3$. Proposition 3 gives $u \neq v$.
2. If either half contains both 0 and 1, then uv is an ordinary square, so $u = v$.
3. Hence, $\{u, v\} = \{0^{|u|}, 1^{|u|}\}$, so z_k contains 000111 or 111000.

Theorem 2: Excluding 000111 and 111000

1. In z_k, 000 can occur only at the start of $\tau(\mathrm{ab})$, and 111 only at the end of $\tau(\mathrm{bc})$.
2. Hence, 000111 or 111000 would require bcab in x_k.

$$\tau(\mathrm{bcab}) = \underbrace{0101}_{\tau(\mathrm{b})}\underbrace{11}_{\tau(\mathrm{c})}\underbrace{00}_{\tau(\mathrm{a})}\underbrace{0101}_{\tau(\mathrm{b})}.$$

Word Construction

$$\varphi : (\mathrm{a}, \mathrm{b}, \mathrm{c}) \mapsto (\mathrm{abc}, \mathrm{ac}, \mathrm{b})$$

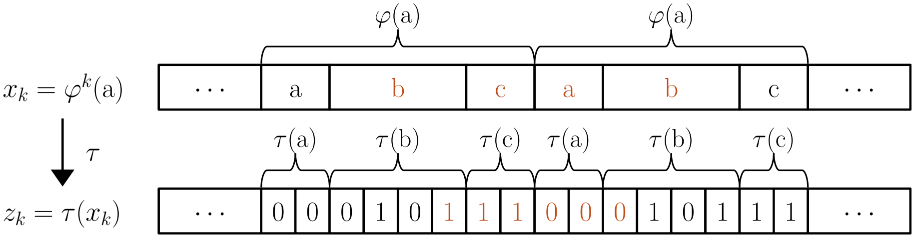
$$\tau : (\mathrm{a}, \mathrm{b}, \mathrm{c}) \mapsto (00, 0101, 11)$$

$$x_k = \varphi^k(\mathrm{a})$$
$$z_k = \tau(x_k)$$

Theorem 2: Excluding bcab

Assume that bcab occurs in x_k.

1. The subwords bc and ab occur only inside $\varphi(\mathrm{a}) = \mathrm{abc}$.
2. Thus, x_k contains the square abcabc, a contradiction.



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Summary

- We construct an infinite 3^+-parameterized-square-free ternary word and an infinite 3^+-order-preserving-square-free binary word using morphisms.
- Computer search gives the maximum lengths in the finite cases.

Strict Equality

$\sigma \backslash \ell$	1	2	3	4
2	3	18	∞	–
3	∞	–	–	–
4	–	–	–	–
5	–	–	–	–

Parameterized Equivalence

$\sigma \backslash \ell$	1	2	3	4
2	1	7	∞	–
3	1	9	–	–
4	1	9	–	–
5	1	9	–	–

Order-Preserving Equivalence

$\sigma \backslash \ell$	1	2	3	4
2	1	7	∞	–
3	1	∞	–	–
4	1	–	–	–
5	1	–	–	–